24th November 2026

Today a paper was announced showing that the 7-hyperbolic manifold M^7 fibres over the circle, with heavy help of AI tools. A few years ago, common practice would have been to share the preprint with people that might be interested in the topic, but alas.

This is particularly demotivating because I was actively working on this with my collaborators Italiano and Ng, and in particular we were exploring the same system of moves used in the preprint. The initial state can be described much better than just a list of 0 and 1: you can start with the state that assigns to $e_i \pm e_{i+1} \pm e_{i+3}$ status 1 (or Out), and to $-e_i \pm e_{i+1} \pm e_{i+3}$ status -1 (or In). Acting with the system of moves where, if you cross a facet F, you invert the status of F and of all facets at distance 2 from it, you obtain a very symmetric orbit, where you have exactly 6 isomorphism classes of ascending links:

1. The initial state is collapsible, and there is a total of 72 ascending links isomorphic to it (including itself);
2. There's another collapsible state with 28 facets, appearing 56 times;
3. One class has homology $\mathbb{Z}$ in dimension 2, has 28 facets and appears 36 times.
4. One class has homology $\mathbb{Z}$ in dimension 2, has 20 facets and appears 28 times,
5. One class has homology $\mathbb{Z}$ in dimension 3, has 28 facets and appears 36 times.
6. One class has homology $\mathbb{Z}$ in dimension 3, has 36 facets and appears 28 times,

This has been checked with a Sage script. Links of type 3. and 5. are the complement of each other, and so are the links of type 4. and 6.

Note that for each pair of antipodal facets, each move inverts the status of exactly one of the two facets. Since the initial state has the property that all antipodal facets have opposite status, applying a move yields a state where antipodal facets have the same status, and viceversa. (The antipodal facet is obtained by inverting all the signs in the octonionic description.) Therefore the links are subdivided in two families, in a checkerboard-like pattern. The family containing the base state consists of exactly the collapsible ones, while the other has all the critical points of index 3 and 4.

We were currently studying the isometry group of the manifold that preserves all the states. There is a very interesing group that acts on the manifold with 256 copies of the polytope. It has a natural subgroup of order 7, acting by cycling the coordinates of the base polytope; we suspect that this acts freely on the links of type 2., 4., and 6. (note that they all appear a number of times that is a multiple of 7). There should be another subgroup of order 9, that acts nicely on the links of the other kind, stabilizing a polytope with ascending link of type 2.

I think all these combinatorial insights should give us a way to concretely perturb the Morse function so that the critical points cancel. I'm happy that the preprint shows that this approach is fruitful, but I would have been happier if the writeup gave more insights on the construction.

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