HYPERBOLIC MANIFOLDS AND FIBRATIONS

MATTEO MIGLIORINI

Abstract. In these lectures, we will study the interplay between hyperbolic manifolds and fibrations (i.e. fibre bundles) over the circle. We will see how on one hand fibering for a hyperbolic manifold is a very weird phenomenon, while on the other hand it is something very common, at least in dimension 3.

Indeed, by a theorem of Agol and Wise, all 3-manifolds virtually fibre over the circle. In higher dimensions the situation is much more mysterious, and the problem is that we lack all the tools we have available in dimension 3 to understand hyperbolic manifolds. The aim of these lectures is to introduce some combinatorial tools that allow to construct hyperbolic manifolds M equipped with a map $f\colon M \to S^1$ that we are able to study.

We start by giving an overview of the world of hyperbolic manifolds, with a particular focus of their relationship with fibrations. We will see how the existence of a hyperbolic manifold of dimension $n > 3$ implies the existence of a hyperbolic group G with a "weird" subgroup H, i.e. that is of finite type but not hyperbolic.

Then we introduce Bestvina-Brady Morse theory, which is a piecewise-linear analogue of the more famous smooth version. This was originally introduced to study the finiteness properties of the kernel of certain epimorphisms $\varphi\colon G \to \mathbb{Z}$; it can be seen as an algebraic analogue of fibrations.

In the appropriate setting one may promote a Bestvina-Brady Brady Morse function to a smooth one. So the problem of constructing a fibration can be reduced to an entirely combinatoric one.

Therefore we introduce Coxeter polytopes, that one can use to construct hyperbolic manifolds equipped with a cell complex structure. Using these, we can construct many different Bestvina-Brady Morse functions that we can investigate with some combinatorial techniques, introduced by Jankiewicz, Norin, and Wise; if some conditions are satisfied, the smoothing will produce a fibration.

1. Lecture 1

1.1. Introduction.

In these lecture series we are going to study two properties of manifolds: the first is admitting a hyperbolic Riemannian metric, and the second one is fibering over the circle. On one hand, these two properties seem to be antithetical: we will see that in a hyperbolic manifold the fibers of a fibration are necessarily very complicated and distorted with respect to the metric; this would suggest that hyperbolic manifolds are likely not to fiber.

On the other hand, in dimension 3 virtually all manifold fiber, while in higher dimensions little is known; this is because there are few tools to construct hyperbolic manifolds and study their topology in dimension > 3. Our objective is to introduce one of these tools, that involves hyperbolic polytopes and a discrete version of Morse theory.

1.2. Fibrations.

Definition 1.1. Let M be a compact manifold with possibly empty boundary. A *fibration over the circle* is a fibre bundle $f\colon M \to S^1$, where each fibre is a properly embedded hypersurface.

All our fibrations will have the circle as base space, so we will just say *fibration*, or that the manifold *fibers*.

Every manifold that fibers over the circle can be fully described by giving the *fibre* and the so called *monodromy*, as follows.

Definition 1.2. Let F be a manifold, and $\varphi\colon F \to F$ be a self-diffeomorphism. The mapping torus $F \rtimes_\varphi S^1$ is defined as

$$F \rtimes_\varphi S^1 := \mathrm{M} \times [0,1] \Big/ (x,1) \sim (\varphi(x), 0)$$

Lemma 1.3. *Suppose that $\varphi, \psi\colon F \to F$ are isotopic. Then $F \rtimes_\varphi S^1 \cong F \rtimes_\psi S^1$.*

Proof: Let $H\colon F \times [0,1] \to F$ be an isotopy between id and $\psi^{-1} \circ \varphi$. Then

$$(x,t) \mapsto (H(x,t), t)$$

is the diffeomorphism. □

The reason behind this notation is that the fundamental group of a mapping torus is $\pi_1(F) \rtimes_{\varphi_*} \mathbb{Z}$.

Lemma 1.4. *Suppose that M fibers over the circle. Then $\chi(M) = \chi(\partial M) = 0$.*

Proof: Assume that M is without boundary, and fix any Riemannian metric on M. Then $\operatorname{grad} f$ is a nowhere vanishing vector field, so $\chi(M) = 0$.

If M has boundary, the conclusion follows by noting that both ∂M and the double DM of M also fiber over the circle, and that $\chi(DM) = 2\chi(M) - \chi(\partial M)$. □

One can also construct a fibration by means of a 1-form.

Proposition 1.5. *Let M be compact manifold, and suppose that α is a nowhere vanishing closed* 1*-form. Assume also that its restriction to the boundary is never vanishing. Then M fibers over the circle.*

Proof: We only prove the case where M has no boundary.

Up to a small perturbation, we may assume that α represents an element in $H^1(M;\mathbb{Q})$. There exists $n \in \mathbb{N}$ such that $n\alpha$ represents an element in $H^1(M;\mathbb{Z})$: now integrating this form yields a map $M \to \mathbb{R}/\mathbb{Z}$ that has no critical points. By using Morse theory, one can conclude that the fibers are all diffeomorphic, so this yields a fiber bundle. □

1.3. The hyperbolic space.

We recall the definition of the hyperbolic space $\mathbb{H}^n$, along with a few properties.

Consider $\mathbb{R}^{n+1}$ equipped with the Lorentzian metric

$$\langle x, y \rangle = x_1 y_1 + \dots + x_n y_n - x_{n+1} y_{n+1}.$$

The hyperbolic space is defined as

$$\mathbb{H}^n = \left\{ x \in \mathbb{R}^{n+1} : \langle x, x \rangle = -1, x_{n+1} > 0 \right\}$$

The restriction of the scalar product to the tangent space of a point in $\mathbb{H}^n$ yields a positive definite bilinear form, so it defines a Riemannian metric.

The above model is inconvenient because we need one extra dimension. To draw pictures, there are some alternatives:

- The Poincaré disk is obtained by projecting on $\{x_{n+1} = 0\}$ through $(0, ..., 0, -1)$. This model is *conformal*, meaning that angles are preserved. Geodesic lines are sent to lines passing through the origin and circles orthogonal to the boundary.
- The half-space model $\{x_n > 0\}$ is obtained by inverting the Poincaré disk at $(0, ..., 0, -1)$. It is also conformal, and geodesic are vertical lines and half-circles orthogonal to the boundary.
- The Klein disk is obtained by projecting on $\{x_{n+1} = 1\}$ through the origin. It is not conformal, but geodesics are Euclidean lines.

1.4. **Hyperbolic manifolds.**

All Riemannian manifolds are assumed to be complete, orientable, with finite volume unless otherwise stated.

Definition 1.6 (Model manifold). A Riemannian manifold is said to be *spherical*, *flat*, *hyperbolic* if it is locally isometric to $S^n, \mathbb{E}^n, \mathbb{H}^n$ respectively.

Definition 1.7. A group (resp. manifold) is said to be *virtually* $\mathcal{P}$ if it has a finite index subgroup (resp. cover) that is $\mathcal{P}$.

Theorem 1.8 (Killing-Hopf). *The universal cover of a spherical, flat, or hyperbolic manifold is respectively S^n, $\mathbb{E}^n$, or $\mathbb{H}^n$.*

In particular, all these manifold are quotient of their model spaces by a discrete group of isometries that acts freely and properly discontinuously.

Corollary 1.9. *All spherical manifolds are compact and virtually a sphere.*

Finite volume flat manifolds are also not that many.

Theorem 1.10 (Bieberbach, 1912). *All finite-volume flat manifolds are virtually an n-torus.*

The world of hyperbolic manifolds is far richer. First of all, there exist non-compact finite volume hyperbolic manifolds. They admit some ends, diffeomorphic to $N \times [0, \infty)$, where N is a flat manifold. Those ends are called *cusps*.

In particular, every finite-volume hyperbolic manifold is diffeomorphic to the interior of a compact manifold with flat boundary components. When we say that a non-compact hyperbolic manifold M fibers, we mean that the compactification $\overline{M}$ fibers.

Remark 1.11. Finite-volume and closed hyperbolic manifolds behave more or less in the same way. For sake of simplicity, most of the theorems will be stated and proved in the closed case, but they can always be generalized to the finite-volume case.

On the contrary, exhibiting example is much easier if we allow cusps.

All finite-volume hyperbolic manifolds are the quotient of $\mathbb{H}^n$ by a discrete subgroup of $\mathrm{Isom}(\mathbb{H}^n) = O(n, 1)$ that acts freely and properly discontinuously.

Proposition 1.12. *A subgroup $\Gamma < \mathrm{Isom}(\mathbb{H}^n)$ acts properly discontinuously if and only if it is discrete.*

Proposition 1.13. *A discrete subgroup $\Gamma < \mathrm{Isom}(\mathbb{H}^n)$ acts freely if and only if it is torsion-free.*

Proposition 1.14 (Selberg's Lemma). *Every finitely generated discrete subgroup of* $\mathrm{Isom}(\mathbb{H}^n)$ *is virtually torsion-free.*

1.5. Isometries of $\mathbb{H}^n$.

We would like to understand better how isomeries act on $\mathbb{H}^n$. We follow [Mar16], Chapters 4–5.

The hyperbolic space, thought as the Poincaré disk, has a natural boundary $\partial\mathbb{H}^n$. Every isometry of $\mathbb{H}^n$ extends uniquely to a homeomorphism of $\overline{\mathbb{H}^n} = \mathbb{H}^n \cup \partial\mathbb{H}^n$.

Isometries of $\mathbb{H}^n$ are all of one of the following three types.

- *elliptic*, if they fix at least one point in $\mathbb{H}^n$;
- *parabolic*, if they do not fix points in $\mathbb{H}^n$ and they fix exactly one on the boundary;
- *hyperbolic*, if they do not fix points in $\mathbb{H}^n$ and they fix exactly two on the boundary.

They can be visualized as follows.

- Elliptic isometries fixing the origin of the Poincaré disk correspond to elements of $O(n)$.
- Parabolic isometries fixing $p \in \partial\mathbb{H}^n$ also preserve all *horospheres* centered at p. They are flat hypersurfaces that are orthogonal to all geodesics exiting from p. In the Poincaré disk model, they are spheres tangent at p. In the half-space model, when p is the point at infinity, the horospheres are horizontal affine hyperplanes.
- Hyperbolic isometries acts as translations along their *axis*, which is the geodesic connecting the two fixed points.

If $\mathbb{H}^n/\Gamma$ is a manifold, then Γ does not contain elliptics. If it is also compact, then it can only contain hyperbolic isometries.

The following tells us that free abelian subgroups are all made of parabolic isometries.

Lemma 1.15. *Let* φ, ψ *be commuting isometries such that* $\langle\varphi, \psi\rangle \cong \mathbb{Z}^2$. *Then they are parabolics fixing the same point* $p \in \partial\mathbb{H}^n$.

This means that hyperbolic manifolds cannot contain essential (i.e. π_1-injective, non-boundary parallel) embedded 2-tori.

Given a discrete subgroup $\Gamma < \mathrm{Isom}(\mathbb{H}^n)$, the boundary at infinity decomposes in two natural subsets.

Definition 1.16. The *limit set* $\Lambda(\Gamma)$ is defined by choosing any base point $x_0 \in \mathbb{H}^n$ and taking

$$\overline{\{\gamma(x_0) : \gamma \in \Gamma\}} \cap \partial\mathbb{H}^n.$$

This does not depend on the choice of x_0. The *domain of discontinuity* is defined by $\Omega(\Gamma) := \partial\mathbb{H}^n \setminus \Lambda(\Gamma)$.

The reason why $\Omega(\Gamma)$ is called domain of discontinuity is that the action of Γ on $\mathbb{H}^n \cup \Omega(\Gamma)$ is properly discontinuous. In particular, we have the following:

Proposition 1.17. *If* Γ *has finite covolume, then* $\Lambda(\Gamma) = \partial(\mathbb{H}^n)$.

Proof: If $x \in \Omega(\Gamma)$, then by the proper discontinuity there is a half-space containing x that intersects only finitely many of its Γ-translates. This implies that Γ has infinite covolume. $\square$

Definition 1.18. A discrete subgroup Γ of $O(n,1)$ is called *elementary* if either of the following equivalent condition holds:

- it stabilizes a finite set of points of $\overline{\mathbb{H}^n}$;
- it stabilizes either a point of $\mathbb{H}^n$, a geodesic line, or a point in $\partial\mathbb{H}^n$ and all the horospheres at that point;
- it is virtually abelian;
- its limit set $\Lambda(\Gamma)$ has at most two points.

Proposition 1.19. *Let Γ be non elementary. Then its action on $\Lambda(\Gamma)$ is minimal, i.e. it does not stabilize any proper closed subset of $\Lambda(\Gamma)$.*

Proof: Let $S \subsetneq \Lambda(\Gamma)$ be a proper closed Γ-invariant subset. Then we can construct the convex hull $C(S)$, given by the intersection of all halfspaces that contain S. Since Γ is non elementary, S has at least two (and actually, infinitely many) points, so $C(S)$ is non-empty.

Clearly, $C(S)$ is Γ-invariant. This implies that $\Lambda(\Gamma) \subseteq S$. $\square$

Corollary 1.20. *Let Γ be non elementary, and $\Gamma' \lhd \Gamma$ be an infinite normal subgroup. Then $\Lambda(\Gamma') = \Lambda(\Gamma)$.*

Proof: By normality, we have that every $\gamma \in \Gamma$ sends $\Gamma'(x)$ to $\Gamma'(\gamma(x))$, so in particular $\Lambda(\Gamma')$ is Γ-invariant. We conclude by the previous proposition. $\square$

2. Lecture 2

We can characterise isometries via their displacement function.

Definition 2.1. The *displacement* of an isometry $\varphi \in \mathrm{Isom}(\mathbb{H}^n)$ is defined by

$$d(\varphi) = \inf_{z \in \mathbb{H}^n} d(z, \varphi(z))$$

Proposition 2.2. *An isometry φ is:*

- *hyperbolic if and only if $d(\varphi) > 0$, and the infimum is realized;*
- *elliptic if and only if $d(\varphi) = 0$ and the infimum is realized on its axis;*
- *parabolic if $d(\varphi) = 0$ and the infimum is* not *realized.*

Let M be a hyperbolic manifold. Denote by $[S^1, M]$ the maps $S^1 \to M$ up to homotopy. These are in natural correspondence to conjugacy classes of $\pi_1(M)$. Note that type (i.e. hyperbolic or parabolic) and displacement is well defined on conjugacy classes of $\pi_1(M)$, that has a natural action on $\mathbb{H}^n$ (up to conjugation).

Proposition 2.3. *Every hyperbolic element of $[S^1, M]$ is represented by an unique closed geodesic with length equal to the displacement.*

A parabolic element of $[S^1, M]$ is never represented by a closed geodesic.

2.1. Hyperbolic surfaces.

In dimension 2, we have a full classification of compact surfaces given the genus g, and the boundary components b.

Theorem 2.4. *The interior of a compact orientable surface S admits a hyperbolic structure if and only if $\chi(S) < 0$, that is, if $2g + b > 2$.*

This tells us that there only finitely many surfaces which are not hyperbolic (the sphere, the disc, the annulus, and the torus). All the other admit a hyperbolic metric (and, in fact, an interesting problem is to study the space of such hyperbolic structures, called the Teichmüller space).

To prove this, we use hyperbolic polygons. We need the following lemma.

Lemma 2.5. *Let $n \geq 3$ be an integer, and let $\alpha \in \left[0, \pi - \frac{2\pi}{n}\right)$. There exists a regular hyperbolic n-gon with angles all equal to α.*

When the angles are 0, the polygon has vertices at infinity; this is called an *ideal polygon*.

Proof: Consider the Poincaré disk in the complex plane, and let $\zeta := e^{\frac{2i\pi}{n}}$ be the n-th root of unity. For $\rho \in (0, 1]$ consider the points $\rho\zeta^k$, for $k \in \{1, ..., n\}$. The convex hull of these points is a hyperbolic n-gon. When $\rho \to 0$, then α tends to the Euclidean angle $\pi - \frac{2\pi}{n}$, while for $\rho \to 1$ then $\alpha \to 0$. We conclude by a continuity argument. □

Remark 2.6. Careful that in dimension ≥ 3 ideal polyhedra have positive dihedral angles!

From this, we get there exists a hyperbolic metric on the pair of pants, by taking two right-angled hexagons and glueing them together.

Proof of Theorem 2.4: One can show that every compact surface S with $\chi(S) < 0$ can be constructed by glueing together $-\chi(S)$ pairs of pants. □

The theorem holds also if we remove some boundary components from S to get cusps. To show this, one can prove that we have the freedom to choose the length on the three boundary components of the pair of pants.

Lemma 2.7. *For every $a, b, c \geq 0$ there exists a hyperbolic right-angled hexagon with alternate side lengths a, b, c. This hexagon is unique up to isometry.*

Corollary 2.8. *For every $a, b, c \geq 0$ there exists a unique hyperbolic metric on the pair of pants such that the boundary are geodesics of length a, b, c. When the length is 0, we mean that the boundary component is replaced by a cusp.*

The Euler characteristic is tightly related with the volume of the manifold – in even dimension, as all closed odd-dimensional manifolds have vanishing Euler characteristic.

Theorem 2.9 (Chern-Gauss-Bonnet). *Let $n \in \mathbb{N}$ be even. There is a constant $c_n \neq 0$ such that for every hyperbolic n-manifold M we have $\chi(M) = c_n \cdot \mathrm{vol}(M)$.*

Proposition 2.10. *Let M be a hyperbolic manifold of even dimension. Then M does not fiber over the circle.*

Proof: Combine Theorem 2.9 with Lemma 1.4. □

Since we cannot obtain fibrations, we can weaken the definition a bit.

Definition 2.11. A *perfect circle-valued Morse function* is a Morse function $f\colon M \to S^1$ with $|\chi(M)|$ critical points.

Proposition 2.12. *Let S be a closed surface of genus g. Then it admits a perfect circle-valued Morse function.*

2.2. Hyperbolic 3-manifolds.

Since we are interested in odd dimensions, the first step is to understand dimension 3. Hyperbolic 3-manifolds are quite important in the study of the topology of 3-manifold, as it is the richest of the eight geometries ($S^3, \mathbb{R}^3, \mathbb{H}^3, S^2 \times \mathbb{R}, \mathbb{H}^2 \times \mathbb{R}, \mathrm{Nil}, \mathrm{Sol}, \widetilde{\mathrm{SL}_2}$) that appear in Thurston's Geometrisation Conjecture, proved by Perelman in 2002.

To study 3-manifold, it is often useful to understand how surfaces are embedded inside it. We introduce the following definitions.

Definition 2.13. Let M be a compact 3-manifold with (possibly empty bondary). A *properly embedded surface* is an embedded surface $S \subset M$ such that $\partial S = \partial M \cap S$.

Definition 2.14. Let M be a compact oriented 3-manifold, and let $S \subseteq M$ be an oriented surface. We say that S is *incompressible* if every loop in S that bounds an embedded disk in M also bounds a disk in S.

Remark 2.15. Being incompressible is equivalent to being π_1-injective, that is the inclusion $S \hookrightarrow M$ induce an injective map between fundamental groups.

Definition 2.16. A properly embedded torus in a compact oriented 3-manifold M is called *essential* if it is incompressible and cannot be homotoped inside ∂M.

Proposition 2.17. *If the interior of M admits a hyperbolic structure, then M does not contain essential tori.*

Let M be a fibering 3-manifold. Assume for the sake of simplicity that it is closed. Let F be a fiber.

Lemma 2.18. *The fiber F satisfies $\chi(S) < 0$.*

Proof: Since all orientation-preserving diffeomorphisms of S^2 are isotopic to the identity, if $F \cong S^2$ then $M \cong S^2 \times S^1$, which cannot admit a hyperbolic structure, since hyperbolic manifolds cannot contain essential tori.

On the other hand, if $F \cong \mathbb{T}^2$, then M contains an essential torus and therefore is not hyperbolic. $\square$

So we can lift to the universal cover

$$\begin{tikzcd}
D \cong \mathbb{H}^2 \arrow[r, hook] \arrow[d, two heads] & \mathbb{H}^3 \arrow[r, two heads] \arrow[d, two heads] & \mathbb{R} \arrow[d, two heads] \\
F \arrow[r, hook] & M \arrow[r, two heads] & S^1
\end{tikzcd}$$

Proposition 2.19. *We have that $\overline{D} \cap \partial\mathbb{H}^3 = \partial\mathbb{H}^3$.*

Proof: The fundamental group $\Gamma := \pi_1(M)$ acts as a group of isometries of $\mathbb{H}^3$, and so does the normal subgroup $\Gamma' := \pi_1(F) \lhd \Gamma$.

It is not difficult to see that $\overline{D} \cap \partial\mathbb{H}^3 = \Lambda(\Gamma')$, which by Corollary 1.20 is equal to $\Lambda(\Gamma)$. Since Γ has finite covolume, the latter is the whole sphere. $\square$

This means that a fiber of a fibering hyperbolic 3-manifold, despite admitting a hyperbolic metric, is very far from being nicely embedded from a metric point of view.

3. Lecture 3

This could suggest that hyperbolic 3-manifolds cannot fiber at all. However, hyperbolic 3-manifold that fiber do exist.

Theorem 3.1 (Jørgensen, 1977). *The complement of the figure-8 knot fibers with fiber the punctured torus.*

Later Thurston gave a criterion for when a manifold admits a hyperbolic structure. First let us see some definitions.

Definition 3.2. A manifold M is said to be:
- *irreducible* if every sphere bounds a ball;
- *atoroidal* if there are no essential tori;
- *Haken* if it contains an incompressible surface.

Theorem 3.3 (Thurston's hyperbolization). *Let M be an irreducible, atoroidal, Haken 3-manifold with toric boundary. Then the interior of M admits a complete hyperbolic metric of finite volume.*

It turns out that we can decide whether a mapping torus admits a hyperbolic metric by looking directly at its monodromy.

Definition 3.4. Let S be a surface of genus ≥ 2, and let φ be a diffeomorphism. We say that φ is:
- periodic, if φ^n is isotopic to the identity;
- reducible, if it preserves a union of finitely many disjoint simple closed curves (up to isotopy);
- pseudo-Anosov otherwise.

Theorem 3.5 (Thurston). *Let S be a closed surface, and $\varphi\colon S \to S$ be a self-diffeomorphism. Then $S \rtimes_\varphi S^1$ is hyperbolic if and only if S has genus ≥ 2 and φ is pseudo-Anosov.*

Proof: One can see directly that it is irreducible and Haken, as the fiber is incompressible (e.g. by looking at the abelian cover). To see that it is atoroidal, one can look at $S \cap T \subset T$. By pushing one can remove non essential curves, and then we are left with a bunch of parallel curves. This means that the fiber cuts the tours in a bunch of annuli, whose boundary curves inside S are preserved by the monodromy; this is a contradiction as we assumed that the map is pseudo-Anosov. $\square$

Later, Agol and Wise managed to prove the following.

Theorem 3.6 (Agol, Wise, ~2010). *Every hyperbolic 3-manifold virtually fibers over the circle.*

3.1. **Higher dimension.**

We have seen that in dimension 3, the fiber admits a hyperbolic structure, even if it does not come from the metric of M. In higher dimension, we also lose this.

Theorem 3.7. *Let M a hyperbolic manifold of dimension $n > 3$, that fibers over the circle with fiber F. Then F cannot admit a hyperbolic metric.*

This is a consequence of the following facts.

Theorem 3.8. *Let M be a hyperbolic manifold. Then the group of isometries* $\mathrm{Isom}(M)$ *is finite.*

Theorem 3.9 (Mostow Rigidity). *Let M be a manifold of dimension ≥ 3. Then every homotopy equivalnce (in particular, every self-diffeomorphism) $f\colon M \to M$ is homotopic to a unique isometry.*

Proof: [Proof of Theorem 3.7] Let $\varphi\colon F \to F$ be the monodromy. If F were hyperbolic, then φ would be isotopic to an isometry, and therefore one of its powers would be isotopic to the identity. This implies that M is virtually a product $F \times S^1$, which contradicts hyperbolicity. □

4. Lecture 4

4.1. Finiteness properties.

Given a fibration $f\colon M \to S^1$, one may pass to fundamental groups and get a map $f_*\colon \pi_1(M) \twoheadrightarrow \mathbb{Z}$. We are interested in the following question: under which assumptions an epimorphism $\varphi\colon \pi_1(M) \twoheadrightarrow \mathbb{Z}$ is induced by a fibration on M? In this sense, the object to study is the kernel of the map and its so called *finiteness properties.*

Definition 4.1. A space X is called *aspherical* if $\pi_n(X)$ is trivial for all $n \geq 2$.

Definition 4.2. Let G be a group. We say that a topological space X is an Eilenberg-MacLane space for G, or that X is a $K(G,1)$, if X is aspherical and $\pi_1(X) \cong G$.

Definition 4.3. A group G is said to be:
- of type $\mathcal{F}_n$ if it admits $K(G,1)$ that is a CW-complex with finite n-skeleton;
- of type $\mathcal{F}_\infty$ if it is of type $\mathcal{F}_n$ for all $n \in \mathbb{N}$;
- of type $\mathcal{F}$ if it is the fundamental group of a finite aspherical CW-complex.

Remark 4.4. A group is of type $\mathcal{F}_1$ if and only if it is finitely generated. It is of type $\mathcal{F}_2$ if and only if it is finitely presented.

Let M be a hyperbolic n-manifold with fundamental group Γ. Its universal cover is $\mathbb{H}^n$, so it is aspherical. Let $f\colon M \to S^1$ be a fibration. We can construct the infinite cover $\widetilde{M} = \mathbb{H}^n / \ker(f_*)$, that is diffeomorphic to a product $F \times \mathbb{R}$.

The cover $\widetilde{M}$ is an infinite volume hyperbolic manifold, and is therefore aspherical, so F also is. This implies that F is an Eilenberg-MacLane space for $\ker(f^*) \cong \pi_1(F)$, so $\pi_1(F)$ is of finite type.

Assume now we have a map $\pi_1(M) \to \mathbb{Z}$. Can it be induced by a fibration? Clearly, we need the kernel to be of type $\mathcal{F}$. In dimension 3, it suffices to check that the kernel is finitely generated.

Theorem 4.5 ([Sta61]). *Let M^3 be a compact 3-manifold with aspherical (possibly empty) boundary. Let $\varphi\colon \pi_1(M) \twoheadrightarrow \mathbb{Z}$ be surjective with finitely generated kernel. Then φ is induced by a fibration.*

For this reason, we have this definition.

Definition 4.6. An epimorphism $\varphi\colon G \twoheadrightarrow Z$ is said to be an *algebraic fibration* if it has finitely generated kernel.

4.2. **Hyperbolic groups.**

Interestingly, constructing fibrations of hyperbolic manifolds has applications in the study of subgroups of hyperbolic groups. Let us recall some definitions.

Definition 4.7. Let X be a metric space. A *geodesic* is an isometric embedding $\gamma : [0, \ell] \to X$, meaning that $d(\gamma(t), \gamma(s)) = |t - s|$. The space is called *geodesic* if for every pair of points there is a geodesic connecting them.

Definition 4.8. Let $G = \langle S \rangle$ be a finitely generated group. Its Cayley graph $\mathrm{Cay}(G, S)$ is the graph whose vertex set is G and there is an arc between g and gs for all $g \in G$ and all $s \in S$.

We can put a metric space structure on the Cayley graph by declaring every edge to have unit length.

Definition 4.9. A geodesic metric space is said to be *δ-hyperbolic* if geodesic triangles are δ-thin, that is, every side is contained in a δ-neighbourhood of the other two.

Example 4.10.
- Every compact space is δ-hyperbolic for δ bigger than the diameter.
- Trees are 0-hyperbolic.
- The Eucledian space $\mathbb{E}^n$ is **not** hyperbolic for $n \geq 2$.
- There exists $\delta > 0$ such that every hyperbolic manifold is δ-hyperbolic.

Definition 4.11. A group G is said to be hyperbolic if it has one of the following equivalent properties:
- its Cayley graph is δ-hyperbolic for some $\delta > 0$,
- it acts properly and cocompactly on a δ-hyperbolic space.

Example 4.12.
- Free groups are hyperbolic.
- A group containing $\mathbb{Z}^2$ is not hyperbolic.
- Fundamental groups of **compact** hyperbolic manifolds are hyperbolic.

Hyperbolic groups satisfy the best possible finiteness properties.

Proposition 4.13. *Every torsion-free hyperbolic group G is type $\mathcal{F}$.*

The proof relies on the construction of the Rips complex.

Definition 4.14. Let X be a metric space, and $R > 0$. The Rips complex $P_R(X)$ is the flag simplicial complex whose vertex set is X and there is an n-simplex for every $(n + 1)$-uple of points in X at pairwise distance $\leq R$.

Lemma 4.15 ([BH99], Proposition 3.23). *Let Y be a geodesic δ-hyperbolic space, and let $X \subseteq Y$ be an r-dense subset. Then $P_R(X)$ is contractible for $R \geq 4\delta + 6r$.*

Proof of Proposition 4.13: Let X be the vertex set of the Cayley graph. By the lemma, for big enough R the Rips complex $P_R(X)$ is contractible. The group G acts on X by isometries, and therefore it acts on $P_R(X)$. This is a finite-dimensional complex, and the action has finite stabilizers; since the group is torsion-free, then this means that the action is free. The quotient is therefore a finite-dimensional $K(G, 1)$. □

Remark 4.16. If G is not torsion-free, one can still get that G is type $\mathcal{F}_\infty$.

Whenever we introduce a property of groups, it is natural to ask whether it passes to subgroups. In general, finiteness properties are not inherited; for example, the commutator subgroup of a free group is not finitely generated. This is also an example of a subgroup of a hyperbolic group that is not hyperbolic, as hyperbolic groups satisfy strong finiteness properties.

It may a priori be possible that if we ask strong enough finiteness properties on the subgroup, we can guarantee its hyperbolicity. However, there are subgroups of hyperbolic groups that are:

- finitely generated not finitely presented [Rip82];
- finitely presented not type $\mathcal{F}_3$ [Bra99];
- type $\mathcal{F}_3$ not $\mathcal{F}_4$ [LMP21];
- more in general type $\mathcal{F}_n$ not $\mathcal{F}_{n+1}$ for all $n \in \mathbb{N}$ [LP22];

So unless we ask that the subgroup is $\mathcal{F}$, we cannot guarantee that it is hyperbolic. By using fibrations, we can show that even that is not enough.

Theorem 4.17. *Let M be a compact hyperbolic n-manifold, $n > 3$, and $F \hookrightarrow M \twoheadrightarrow S^1$ be a fibration. Let $G := \pi_1(M)$, $H := \pi_1(F)$. Then G is hyperbolic, and H is a subgroup that is type $\mathcal{F}$ but not hyperbolic.*

Proof: G is hyperbolic since it is the fundamental group of a compact hyperbolic manifold, and F is of type $\mathcal{F}$ since it is the fundamental group of a compact manifold. We only need to show that F is not hyperbolic, which is the algebraic translation of Theorem 3.7.

The monodromy $\varphi\colon F \to F$ induces an automorphism $\varphi_*\colon H \to H$. This has infinite order in $\mathrm{Out}(H)$ (otherwise G would be virtually $H \times \mathbb{Z}$).

By Rips' theory, every hyperbolic group with infinite outer automorphism group splits over a cyclic subgroup. So, if H were hyperbolic, then $H = A *_{\mathbb{Z}} B$.

Then we have the following Mayer-Vietoris sequence:

$$H^{n-2}(\mathbb{Z}) \to H^{n-1}(H) \to H^{n-1}(A) \oplus H^{n-1}(B) \to H^{n-1}(\mathbb{Z})$$

Since $n > 3$, we get $H^{n-1}(H) \cong H^{n-1}(A) \oplus H^{n-1}(B)$.

However, both A and B are infinite index subgroups of H, and therefore they are the fundamental group of some non-compact $(n-1)$-manifold. So their $(n-1)$-th cohomology vanishes, while $H^{n-1}(H) \cong \mathbb{Z}$. □

Note that a similar result can also be obtained when M is only finite volume. In this case, one must first perform some sort of *filling* (otherwise the fundamental group would not be hyperbolic).

5. Lecture 5

5.1. Bestvina-Brady Morse Theory.

Now we have motivated why constructing high dimensional hyperbolic manifolds that fiber is an interesting problem. How one can produce such examples?

First of all, constructing hyperbolic manifolds in high dimension is not easy. Even the existence of hyperbolic manifolds in all dimensions is not a trivial fact.

There are two known methods to build hyperbolic manifolds.

- The first one is via arithmetic methods. More or less, the idea is to look at integer lattices of $O(n, 1)$ to construct discrete subgroups of $\mathrm{Isom}(\mathbb{H}^n)$. This is a very reliable method to build hyperbolic manifolds, but has the disadvantage that we have poor understanding of the topology of such manifolds.
- The second one is combinatorial, by glueing together copies of polytopes. This is less reliable, as we will see that it cannot produce examples in arbitrary dimensions, but is more concrete.

We will focus on the second approach. To this end, let us introduce some notion of piecewise linear geometry. We follow [RS82].

Definition 5.1. A *polyhedron* is a subset $X \subseteq \mathbb{R}^d$ such that every point $a \in X$ has a cone neighbourhood of the form aL. The neighbourhood is called a *star* of p, while L is called a *link*.

Definition 5.2. A map $f\colon X \to Y$ between polyhedra is *piecewise linear* or *PL* if every $a \in P$ has a star on which f is linear along rays, i.e. $f(\lambda a + \mu x) = \lambda f(a) + \mu f(x)$.

Definition 5.3. A PL-structure on a topological set X is an atlas such that charts map to polyhedra, and transition maps are PL.

We can always assume that the link and star of a point are polyhedra; in this case, they are well-defined up to PL isomorphisms.

Definition 5.4. An *affine polytope* is the compact intersection of finitely many half-spaces of $\mathbb{R}^d$. Equivalently, it is the convex hull of finitely many points in $\mathbb{R}^d$.

A *face* of a polytope P is the intersection of P with the boundary of a hyperplane containing it. The empty set and P itself are conventionally considered facets.

Facets, *ridges*, *edges*, and *vertices* are respectively 1-codimensional, 2-codimensional, 1-dimensional, and 0-dimensional faces of P.

Definition 5.5. An *affine cell complex* is a CW-complex X, together with a characteristic function $\chi_\sigma\colon \sigma \to \mathbb{R}^d$ from each cell σ of X, such that:

- each χ_σ is a homeomorphism on its image;
- whenever $\tau < \sigma$, then the composition $\chi_\sigma \circ \chi_\tau^{-1}$ is affine.

Example 5.6.

- Every simplicial complex is naturally an affine cell complex, with the characteristic function sending each n-simplex to the standard n-simplex.
- A cube complex is an affine cell complex with the characteristic function sending n-cubes to $[0, 1]^n$.

We introduce a piecewise linear version of Morse theory, that was introduced by Bestvina and Brady.

Definition 5.7 ([BB97]). Let X be an affine cell complex. A real-valued *Bestvina-Brady Morse function* is a map $f\colon X \to \mathbb{R}$ such that, for each positive-dimensional cell $\sigma \in X$:

- the composition $f \circ \chi_\sigma^{-1}$ is the restriction of an affine map $\mathbb{R}^d \to \mathbb{R}$;
- the restriction $f|_\sigma$ is nonconstant.

Similarly, a circle-valued Bestvina-Brady Morse function is a map $f\colon X \to S^1$ such that its lift $\tilde{f}\colon \tilde{X} \to \mathbb{R}$ is Bestvina-Brady Morse.

In what follows, given a Bestvina-Brady Morse function $f\colon X \to \mathbb{R}$ and $a \in \mathbb{R}$, we denote by $X_{\leq a}$ the sublevel $f^{-1}(-\infty, a]$. Bestvina-Brady Morse functions behave similarly to standard Morse function: the topology of sublevels may change only when crossing a vertex.

Definition 5.8. Let $Y \subseteq X$ be polyhedra, and let $B \subseteq P$ such that $Y \cup B = X$ and $B \cap Y$ is a $(d-1)$-ball. In this case, we say that there is an elementary collapse $X \searrow\!\!\searrow Y$. We say X collapses to Y, or $X \searrow Y$, if there is a sequence of elementary collapses from X to Y. If X collapses to a point, we say that X is collapsible.

Lemma 5.9. *Let $f\colon X \to \mathbb{R}$ be Morse, and let $a < b$ be such that the subset $f^{-1}[a, b)$ does not contain any vertex of X. Then $X_{\leq b}$ collapes onto $X_{\leq a}$.*

Proof: Subdivide the cell complex by cutting along the levels a and b. For each cell σ, starting from the highest-dimensional ones, we can perform an elementary collapse by removing the interior of σ and of $\sigma \cap f^{-1}(b)$. □

What happens when we cross a vertex? To describe it, we need the definition of ascending and descending link.

Definition 5.10. The *ascending link* of a vertex v, denoted $\mathrm{link}_\uparrow(v)$, is the intersection of $\mathrm{link}(v)$ with cells where v is a minimum for f. Similarly, the *descending link*, denoted $\mathrm{link}_\downarrow(v)$, is the intersection of the link with cells where v is a maximum.

Theorem 5.11. *Let $f\colon X \to \mathbb{R}$ is a real-valued Morse function on an affine cell complex X, and let $a < b \in \mathbb{R}$ such that there is a single vertex v with $a < f(v) \leq b$. Then $X_{\leq b}$ collapses to $X_{\leq a}$ with a cone attached over $\mathrm{link}_\downarrow(v)$, where $\mathrm{link}_\downarrow(v)$ embeds naturally inside $f^{-1}(a)$.*

Proof: Collapse all cells that intesect $f^{-1}(b)$ in a top-dimensional cell. After that, we are left with $X_{\leq a}$ with a cone attached over $\mathrm{link}_\downarrow(v)$. □

5.2. **Right-angled Coxeter groups and polytopes.**

Let Γ be a finite simplicial graph.

Definition 5.12. The right-angled Coxeter group W_Γ is the group with the following presentation:

$$W_\Gamma := \langle V(\Gamma) \mid v^2 = 1 \text{ for } v \in V(\Gamma), [v, w] = 1 \text{ for } (v, w) \in E(\Gamma)\rangle.$$

If Γ' is a subgraph of Γ, one may consider the subgroup generated by the vertices of Γ': this is canonically isomorphic to $W_{\Gamma'}$.

Proof: Let H be the subgroup generated by the vertices. We have the following commutative diagram:

$$\begin{tikzcd}
H \arrow[r, hook] & W_{\Gamma} \arrow[d, two heads] \\
& W_{\Gamma'} \arrow[ul, two heads]
\end{tikzcd}$$

and the composition $H \to W_{\Gamma'}$ is surjective. □

The group W_Γ acts on a canonical space, called the Davis complex. To construct it, we consider the poset $\mathcal{P}$ given by finite cosets of the form $gW_{\Gamma'}$, so Γ' is a clique in Γ. The poset is also graded by the number of generators of $W_{\Gamma'}$. Then we can construct the Davis complex by considering the geometric realisation of $\mathcal{P}$ as simplicial complex.

We will denote the Davis complex with D_Γ. W_Γ acts properly and cocompactly by left multiplication.

The interesting case for us is when this group comes from reflections on the faces of a hyperbolic polytope.

Definition 5.13. A *right-angled hyperbolic polytope* $P \subseteq \mathbb{H}^n$ is the finite-volume intersection of finitely many half-spaces, such that every pair of hyperspaces is either disjoint or it intersects orthogonally.

The adjacency graph of P is the graph whose vertices correspond to facets of P, and an edge connects two facets that intersect in a ridge.

If we take the flag complex of the adjacency graph, one gets a polytope called the dual of P.

Reflecting on the facets of P defines an action of W_Γ on $\mathbb{H}^n$, and yields a tessellation of $\mathbb{H}^n$ into copies of P.

Proposition 5.14. *The barycentric subdivision of the tessellation is isomorphic to the Davis complex of W_Γ, with W_Γ acting equivariantly.*

Proof: Note that the Davis complex is a manifold (one may check the links of the points, they are the dual of P). Now we define a map from D_Γ to C as follows: for each vertex $gW_{\Gamma'}$, consider the face F of P stabilized by $W_{\Gamma'}$, which is the intersection of the facets corresponding to vertices of Γ'. We send $gW_{\Gamma'}$ to the barycentre of F.

This map is a well-defined covering map, so it is a homeomorphism since $\mathbb{H}^n$ is simply connected. □

Instead of considering the tessellation into copies of P, it is more useful to look at the *dual tessellation*. This is a cube complex, since we only have right angles. Note that the barycentric subdivision of this cube complex is still isomorphic to the Davis complex.

Remark 5.15. When P has ideal vertices, one has to be a bit careful about the definition of barycentric subdivision: this only yields a retract of the original tessellation.

6. Lecture 6

Let P be a hyperbolic right angled n-polytope. Let Γ be the associated adjacency graph. The associated reflection group is isomorphic to the Coxeter group W_Γ, and generates a tessellation of $\mathbb{H}^n$; last time we saw that the barycentric subdivision is isomorphic to the Davis complex.

We recall how this identification works. We fix one copy P_e of the polytope. Every other polytope is of the form gP_e for a unique $g \in W_\Gamma$; we call it therefore P_g. The action is then defined by $h \cdot P_g = P_{hg}$.

It is not difficult to show that a k-codimensional cell σ belongs to 2^k copies of P; these are of the form $\{P_h : h \in gW_{\Gamma'}\}$, where Γ' is a clique of Γ. In particular, an element $h \in W_\Gamma$ fixes σ if and only if it stabilizes the coset $gW_{\Gamma'}$, that is $g^{-1}hg \in W_{\Gamma'}$.

We now want to construct a hyperbolic manifold by considering an appropriate quotient of this tessellation. That is, we need to find $G < W_\Gamma$ that acts freely and such that the quotient has finite volume; that is, we need that G has finite index in W_Γ.

6.1. **The Löbell construction.**

Clearly W_Γ is not torsion-free (it has fixed points). Fortunately, for right-angled Coxeter groups we have a standard torsion-free subgroup.

Proposition 6.1. *The commutator subgroup $G < W_\Gamma$ is torsion-free.*

Proof: We show that it does not contain any elliptic. The stabilizer of $gW_{\Gamma'}$ is the subgroup $gW_{\Gamma'}g^{-1}$. Since $W_{\Gamma'}$ survives in the abelianization, its intersection with G is trivial. ☐

So $M := \mathbb{H}^n / G$ is a hyperbolic manifold. This is tessellated by 2^c copies of P, where c is the number of facets of P: they are in correspondence with $W_\Gamma / G = (W_\Gamma)_{\mathrm{ab}} = (\mathbb{Z}/2\mathbb{Z})^c$.

Remark 6.2. The manifold can be constructed iteratively by doubling along facets.

The dual of this tessellation is a cube complex C, that can be visualized as follows. The vertices of the cube complex are in 1–1 correspondence with copies of the polytope, i.e. with $(\mathbb{Z}/2\mathbb{Z})^c$ (after fixing a numbering for the facets). These are naturally the vertices of a c-cube.

The 1-skeleton is naturally the 1-skeleton of the cube. Then, whenever we see the 1-skeleton of a k-cube whose corresponding k-facets are pairwise adjacent.

Lemma 6.3. *Let v be a vertex of C. Then* $\mathrm{link}(v) \cong \Delta(\Gamma)$*, where* $\Delta(\Gamma)$ *is the unique flag complex with* Γ *as* 1*-skeleton.*

Recall that if P is a compact polytope, then this cube complex is PL-homeomorphic to a hyperbolic manifold M.

We want to define a circle-valued Bestvina–Brady Morse function on C. We do this inductively on the skeleton.

First, we let $f(v) = 0 \in {}^{\mathbb{R}}/_{\mathbb{Z}}$ for every vertex $v \in C$. Then, we want to extend it on every edge so that on each it does a full turn around S^1. To do so, we need to choose an orientation on each edge.

We use the following combinatorial object, introduced by [JNW19].

6.2. **States.**

Let v be a vertex of C.

Definition 6.4. A *state* on v is a labelling of the vertices of Γ with I (for In) and O (for Out).

So we think of it as a function $s_v\colon (\mathbb{Z}/2\mathbb{Z})^c \to \{O, I\}$. This induces an orientation on all the edges incident to v.

The plan is defining a state on every vertex of C, so that they define an orientation on every edge. There is one caveat: the orientation induced by the two endpoints of an edge should be the same. That is, we need

$$s_{v+e_i}(i) \neq s_{v(i)}$$

for all $i \in \{1, ..., c\}, v \in (\mathbb{Z}/2\mathbb{Z})^c$.

Instead of defining a state on every vertex and check this condition, we choose an initial state s_0 and we propagate it with *moves*.

6.3. **Moves.**

Definition 6.5. A *move* is a subset m of the facets. A *set of moves* is a partition of the facets into moves.

A move m acts on a state by inverting the status of all facets in m, and leaving the other unchanged.

Choose a set of moves. For $i \in \{1, ..., c\}$, we let m_i be the move containing the i-th facet. We now define

$$s_{e_{i_1}+...+e_{i_k}} = m_{i_1}\Big(...\Big(m_{i_k}(s_0)\Big)...\Big).$$

Lemma 6.6. *We have that* $s_{v+e_i}(i) \neq s_{v(i)}$.

Proof: Since the actions of the moves commute, we have that $s_{v+e_i} = m_{i(s_v)}$, and $m_{i(s_v)}(i) \neq s_{v(i)}$ by definition. □

This defines an orientation of each edge of C, and a map $C^{(1)} \to S^1$ given by sending each edge, identified as the interval $[0, 1]$, to S^1 via $[0, 1] \hookrightarrow \mathbb{R} \twoheadrightarrow {}^{\mathbb{R}}/_{\mathbb{Z}}$.

We now try to extend it to the 2-skeleton. We analyze the configurations that may appear: we look at a square, that corresponds to a pair of adjacent facets F_i, F_j.

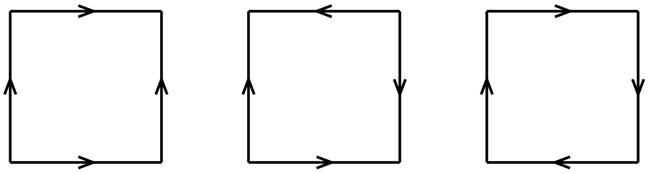

FIGURE 1. The three possible configurations of squares that may appear. The first is when F and F' belong to different moves. The second is when F, F' belong to the same move, but they have the same status. The third is if they belong to different moves, and they have opposite status.

Definition 6.7. A set of moves is called *sparse* if each pair of adjacent facets belongs to different moves.

In this case, all the squares are of the first kind. We extend the map f to the whole C as follows. Let Q be a k-cube of C; this is canonically identified with $[0,1]^k$ (up to a permutation of the coordinates). We define on it $f\colon Q \to S^1$ by

$$f(x_1, ..., x_n) = x_1 + ... + x_n \operatorname{mod}(\mathbb{Z}).$$

This defines a Bestvina–Brady Morse function on C. We want to study the ascending and descending links of this function.

Proposition 6.8.

- *If all the ascending and descending links are connected, then f is an algebraic fibration.*
- *If all the ascending and descending links are collapsible, then f can be smoothened to a smooth fibration over the circle.*

Ascending and descending links can be studied well with the states. If s is a state, denote by Γ_s^O and Γ_s^I the induced subgraphs of Γ with labels O and I respectively.

Lemma 6.9. *The ascending and descending links at v are $\Delta\left(\Gamma_{s_v}^O\right)$ and $\Delta\left(\Gamma_{s_v}^I\right)$ respectively.*

This tells us that, if we find a sparse set of moves and an initial state, such that its orbit with respect to the action of the moves has all collapsible ascending and descending links, then we are done.

However, the requirement of being sparse is very strong, so it is difficult to find this property on high-dimensional right-angled hyperbolic polytopes.

We relax the requirement to allow the second type of square. The third is not allowed as we cannot extend the function continuously to the whole square.

Definition 6.10. A state is *compatible* with the set of moves if, whenever two adjacent facets are in the same move, they have the same status.

When the initial state is compatible, there is still a way to define a Bestvina–Brady Morse function, but this time we need to perform a subdivision (one barycentric subdivision is enough). This allowed to prove the following.

Theorem 6.11 ([IMM22]). *There exists a hyperbolic 5-manifold that fibers over the circle.*

References

[Mar16] B. Martelli, “An Introduction to Geometric Topology,” *arXiv:1610.02592 [math]*, Oct. 2016, [Online]. Available: http://arxiv.org/abs/1610.02592

[Sta61] J. R. Stallings, “On fibering certain 3-manifolds,” 1961. [Online]. Available: https://api.semanticscholar.org/CorpusID:118959999

[BH99] M. R. Bridson and A. Haefliger, *Metric Spaces of Non-Positive Curvature*, vol. 319. in Grundlehren der mathematischen Wissenschaften, vol. 319. Springer, 1999. doi: 10.1007/978-3-662-12494-9.

[Rip82] E. Rips, “Subgroups of small cancellation groups,” *Bulletin of the London Mathematical Society*, vol. 14, pp. 45–47, 1982, doi: 10.1112/blms/14.1.45.

[Bra99] N. Brady, “Branched Coverings of Cubical Complexes and Subgroups of Hyperbolic Groups,” *Journal of the London Mathematical Society*, vol. 60, pp. 461–480, 1999, doi: 10.1112/S0024610799007644.

[LMP21] C. Llosa Isenrich, B. Martelli, and P. Py, “Hyperbolic groups containing subgroups of type \mathcal{F}_3 but not \mathcal{F}_4”, *arXiv:2112.06531*, 2021.

[LP22] C. Llosa Isenrich and P. Py, “Subgroups of hyperbolic groups, finiteness properties and complex hyperbolic lattices,” *arXiv:2204.05788*, 2022.

[RS82] C. P. Rourke and B. J. Sanderson, *Introduction to Piecewise-linear Topology.* in Ergebnisse der Mathematik und ihrer Grenzgebiete. Springer-Verlag, 1982. [Online]. Available: https://books.google.it/books?id=AoEpAQAAMAAJ

[BB97] M. Bestvina and N. Brady, “Morse theory and finiteness properties of groups,” *Inventiones mathematicae*, vol. 129, pp. 445–470, 1997.

[JNW19] K. Jankiewicz, S. Norin, and D. T. Wise, “Virtually fibering right-angled Coxeter groups,” *Journal of the Institute of Mathematics of Jussieu*, vol. 20, pp. 957–987, 2019, doi: 10.1017/s1474748019000422.

[IMM22] G. Italiano, B. Martelli, and M. Migliorini, “Hyperbolic 5-manifolds that fiber over S^1”, *Inventiones mathematicae*, vol. 231, pp. 1–38, 2022.